THE PSYCHOLOGY OF MATHEMATICS FOR INSTRUCTION

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LAWRENCE ERLBAUM ASSOCIATES, PUBLISHERS
1981 HILLSDALE, NEW JERSEY

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7 Piaget and the Development of Cognitive Structures

We turn now to the work of Jean Piaget and a somewhat different view of cognitive structures than that conveyed by the gestalt movement. Gestalt psychologists, as we have seen, focused particularly on the immediate way in which structures of problems or of subject matters were perceived, as if entire structures were taken in "at a glance." Because of its emphasis on the immediacy of insight and the relatively complete understanding that usually ensued, gestalt psychology seemed unconcerned with how knowledge of relationships built up to the point where such insight and recognition were possible. Nor did the gestaltists seem concerned with how, over extended periods of time, people's capacities for recognition and insight might change. In contrast, Piaget was explicitly concerned with the process and development of thinking. He also believed that the fundamental characteristics of human thinking could be understood in terms of the logical propositions and relationships that human behavior expressed. Both his interest in logic and his concern with how thinking is modified during growth and experience helped to shape his definition of cognitive structure.

Piaget is best known for his extensive studies of the development of children's thinking. Most discussions of his work particularly emphasize the idea of stages of development, and many summaries of Piaget's theory outline the sequence of stages that he proposed in the course of his research. We outline these stages and consider what they imply for teaching mathematics to children. However, we begin our discussion of Piaget with an attempt to understand his view of the role of structure in thinking. Later, we review some of the major alternative interpretations of intellectual development that have been proposed by "neo-Piagetians" and some anti-Piagetians. We also consider and evaluate the various instructional

implications that are often drawn from Piaget's work, particularly those related to mathematics.

THINKING AS STRUCTURING

Piaget began his scholarly career as a biologist, and this orientation has permeated virtually all his work in psychology as well. As a biologist, he was interested in the physical structures that characterized organisms. He noted that those structures underwent a gradual development over generations so as to make the organisms better adapted to their environments. As a psychologist, Piaget was interested in cognitive structures—the structures of thinking. Although cognitive structures could not be observed directly, as physical ones could, Piaget tried to reveal the thought processes of children through a technique of activity-based interviewing. His style of interviewing, displayed in sample protocols throughout this chapter, provides an important alternative method of investigation for psychologists and educators interested in probing the nature of individuals' thinking and understanding. Although Piaget's research is based exclusively on clinical interviewing, the greatest value of the method to psychology probably lies in its judicious combination with other more experimental methods, much as combining protocol analysis with more quantitative research strategies is proving profitable in information-processing research.

Much of Piaget's work was premised on the notion that individuals recapitulate, in the course of their development, the intellectual history of the human species. Piaget thought it possible, therefore, to understand the development of the species' intellectual capacities by studying the intellectual development of individuals as they grew into adults. As Piaget's work progressed, he became increasingly convinced that certain basic structures of thinking, which could be defined logically and mathematically, were inherent for human beings. By inherent, Piaget did not mean that people were born with these structures fully formed or that children raised apart from normal human relationships would develop them. He meant, rather, that all humans would develop certain structures of thinking as long as they maintained a normal interaction with both the social and physical environment. The idea was that people were biologically constructed to interact in certain ways with their environment. In the course of this interaction a sequence of complex structures of thinking would emerge.

Although the development of children's thinking could be studied in many subject areas, Piaget's most extensive work was on the development of logical and mathematical concepts. He studied, in both children and adolescents, the growth of logical classification systems and the concepts of number, geometry, space, time, movement, and speed. These topics were chosen for intensive study because they clearly involved the use of certain basic logical structures. Along with earlier philosophers who studied epistemology—the science of

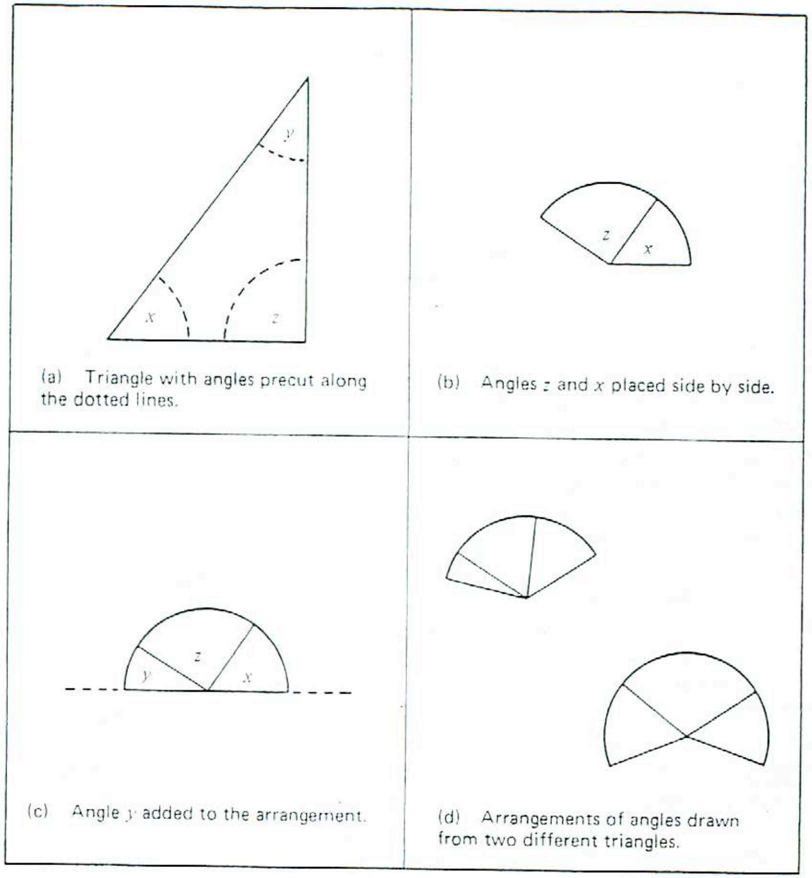
knowledge—Piaget believed these structures were the basis of thinking and reasoning, particularly of a scientific kind. We can best convey Piaget's notion of structures by considering examples from his research. We begin with some experiments on children's conceptions of geometry (Piaget, Inhelder, & Szeminska, 1948/1960).

The Angles of a Triangle

A child is shown a triangle, and it is cut, as shown in Fig. 7.1a, so that the three angles can be picked up and handled separately. Two of the angles are arranged side by side (Fig. 7.1b). The child is asked to predict what they will look like when the third angle is added. After the prediction, the third angle is added and the result is a half-moon (Fig. 7.1c); that is, the angles sum to 180 degrees, or form the equivalent of a straight line. Will this be the case for the sum of the angles of all triangles?

The point of interest is not what the child actually knows about the angles of a triangle at the beginning of the experiment but how the child thinks about the problem during the experiment, and whether by the end he or she becomes quite convinced that the three angles put together will always form a straight line (i.e., that they will always sum to 180 degrees). To find out how the child is thinking, the experimenter presents the problem several times using triangles of different shapes and sizes. Sometimes, after the angles have been arranged in a line, the experimenter rearranges their order and asks the child to predict whether they will still fit along the line. This is a way of checking whether the child understands that the order of angles does not affect their sum. At other times the experimenter tries to fool the child by taking the third angle from a different triangle and asking how it will look alongside the two angles from the first triangle. Or, late in the experiment, after the child appears convinced that the angles will always sum to 180 degrees, the experimenter introduces a set of angles summing to more or less than 180 degrees and asks how this could be (see Fig. 7.1d). A child whose understanding of the nature of a triangle is secure is likely to guess that the pieces were taken from more than one triangle. What happens under these various conditions, and how are Piaget's structures of thinking revealed?

In the following paragraphs we discuss samples of verbal behavior elicited by Piaget while studying children's conceptions of geometry. The sample protocols are shown in accompanying figures. The protocols typify the kind of dialogue between experimenter and subject that has characterized Piaget's clinical style of research over the years. Individual children are presented a problem or situation and are asked to verbalize their thinking as they proceed; the experimenter questions and probes each child's thinking, pursuing thoughts in the order and depth suggested by children's verbalizations. Questioning is led, in part, by the moment-to-moment responses of the children.



(a) Triangle with angles precut along the dotted lines.

(b) Angles z and x placed side by side.

(c) Angle y added to the arrangement.

(d) Arrangements of angles drawn from two different triangles.

FIG. 7.1 The angles-of-a-triangle problem. Some children discover that the angles of a triangle always sum to 180°.

Some children, usually the 5- and 6-year-olds, are quite unable to treat the angles-of-a-triangle problem as anything but a string of unrelated guesses. They make predictions but apparently without system and without justification. Figure 7.2 gives an example of such a child, Ful. At first, Ful cannot make a prediction at all (Fig. 7.2a), although he can recognize and name a half-moon after the angle pieces are put together. When asked for a prediction on a second triangle, he appears to have generalized from this first experience, because he predicts the angles will form a half-moon again (b). But we soon see that he has not really understood the principle: He predicts a sum of less than 180 degrees for the angles of the next triangle (c) and gives no reason for his prediction (d). He also

EXPERIMENTER	FUL
(a) Presents right angle triangle with precut angles.	Puts together three precut angles taken from the right angle triangle and sees that they form a semi circle although he failed to predict it.
(b) Presents another triangle. *What if I put these two corners together and then put the third corner alongside, what will these make?*	*They'll make a half-moon as well.*
(c) Shows another right angle triangle, much larger. *What if I do the same with the corners of that triangle?*	Indicates a figure less than 180°.
(d) *Why will it be less?*	Doesn't reply. Experiments with the pieces. *Oh, it's the same!*
(e) *Now what if I take this corner on this side and put it over on that side instead of this one and put this one over here?* Indicates an exchange of pieces as he speaks.	*It'll be bigger.*
(f) *Are you sure?*	*No, it'll be half-moon.*
(g) *Why?*	*No, it will be more.*
(h) Presents a trapezium (four-sided figure). *And what if we put the corners of this figure together?*	*They'll make a whole circle because there are several large ones.*
(i) Presents a small equilateral triangle. *And the three corners of this one?*	*They'll make a half-moon and a little bit.*
(j) *Why?*	*These two little corners make a half-moon.* Experiments. *Oh, no! It's a half-moon.*
(k) *And if I change these two?* Indicates exchange as before.	*A half-moon leaning over. . . I don't know.*

FIG. 7.2 Experimenter interviewing child working on the angles-of-a-triangle problem. Ful, 5½ years old, demonstrates an incomplete understanding of the problem. (Adapted from Piaget, Inhelder, & Szeminska, 1960. Copyright 1960 by Routledge & Kegan Paul, published in the U.S. by Basic Books, Inc.)

thinks that rearranging the pieces will make the set larger (e). He is easily influenced by the experimenter, for when asked if he is sure about the rearranged pieces (f), he reverts to the half-moon prediction. Then, in response to the experimenter's "Why?," he says it will be more (g). Apparently, Ful has no strong idea of his own and is just following what he thinks are the experimenter's clues. When a four-sided figure, the trapezium, is presented, Ful makes a correct prediction (h), but we cannot tell whether he recognizes that the angles of four-sided figures will always sum to 360 degrees. Probably he does not, for on the next triangle (i, j) he is estimating again, apparently from perceptual features; this time he thinks two of the corners will make a half-moon by themselves, and the third will make "a little bit" more. And once again (k) he thinks reversing position might change quantity, but he is not sure.

Compare Ful's performance with that of another child, Jeq, 5 years older. His responses appear in Fig. 7.3. Although Jeq has some trouble with verbalization, he nevertheless makes it clear (b) that he is looking at the three angles in relation to one another and thinking about how the angles and directions of the lines in the figure are related. He sees that if the lines are to meet, it is necessary that the angles sum to only 180 degrees. If they sum to more than that, there must be four or more angles; that is, the figure must be some shape other than a triangle (b, d). Reluctant at first to predict the sum of angles, Jeq proceeds to make a prediction before he has finished assembling the cut angles (c). Then he gets the idea that the angles will always form a semicircle (c) and sets out to test (d) what he is quite obviously treating as a hypothesis.

Jeq indicates at several points that he is considering the angles in relation to each other: "One big and two thin" (d), "These two angles aren't quite right angles, they're made up by the one at the top" (e). In other words, he sees the triangle not just as a given perceptual form but as an interrelated system of lines and angles. He recognizes, too, that modifying any one part of the system will require a compensating modification in some other part. Thus, the enlargement of one of the angles must be offset by a reduction in one or more of the other angles if the system is to remain intact, that is, if it is to remain a triangle.

A comparison of the performances of Jeq and Ful highlights Piaget's use of the term *structure*. Jeq responds to the triangle in terms of interrelated features—lines and angles—and sees that an action in any part of the triangle will change the whole figure. In other words, Jeq understands the figure in terms of relationships he creates through his thought processes; he conceives the figure as a whole, features of which can be thought of independently. Ful, on the other hand, is not able to think of all of these relationships simultaneously. He does sometimes try to consider more than a single feature at a time. But he is not very good at it, apparently because he does not see any connection between the lines and the angles of the triangle. He is thus reduced to guessing and perceptually approximating. His analysis of the problem is unsystematic, and he lacks confidence in his judgments.

EXPERIMENTER	JEQ
(a) Presents right angle triangle with precut angles.	Assembles three angles. *A semi circle.*
(b) *Why?*	*I can't tell you, it's because the angles are smaller than they would be if they were right angles. . . . You'd need four; three angles wouldn't meet* (if they were greater than 180°).
(c) Presents another triangle. *And that one?*	*I can't say in advance.* Prepares to assemble the angles. *It's another semi circle. I'm beginning to think it's always a semi circle. With three angles you can't have a full circle because you'd need angles bigger than a right angle; you'd need a further stroke* (i.e., side).
(d) Presents an isosceles triangle with a very obtuse angle.	*One big and two thin; a semi circle again! I want to try again with that one* (an even more obtuse-angled triangle). *Yes, still the same because there's an enormous angle at the top.*
(e) Presents very pointed isosceles triangle. *And this one?*	*A semi circle again! These two angles aren't quite right angles, they're made up by the one at the top. Two right angles would make a semi circle?*
(f) *And if we made it so long that it reached the cellar?*	*It's always the same if the lines are quite straight.*

FIG. 7.3 Protocol of an 11-year-old child, Jeq, successfully attempting to understand the angles-of-a-triangle problem. (Adapted from Piaget, Inhelder, & Szeminska, 1960. Copyright 1960 by Routledge & Kegan Paul, published in the U.S. by Basic Books, Inc.)

Another difference between the two performances is that Jeq is quite clearly able to make predictions and test his ideas, although we have no evidence that his tests are fully systematic and exhaustive of all possibilities. By contrast, Ful does not formulate or test a general hypothesis that he thinks will apply to all triangles. Rather, his actions seem to respond solely to the experimenter's cues. Thus building structure in the Piagetian sense appears to involve constructing relationships such that change in any part of the system affects the whole system. Further, the more advanced forms of structuring lead to hypotheses about general relationships, thus freeing thought from the immediate stimulus at hand.

Matching Spatial Orientation in a Coordinate System

Let us consider another example. The task is deceptively simple: the child is shown a rectangle with a dot in it and another identical rectangle with no dot. The situation is depicted in Fig. 7.4a. The child is then asked to place a dot in the empty rectangle so that it matches exactly the position of the dot in the first rectangle. Imagine for a moment how an adult might try doing the task. Perhaps one's first inclination would be to estimate visually; no doubt one could match the position quite well by this method. But the experimenter in this study does not accept estimation. An exact match of position is required. How could the position of the dot be located exactly?

Here is what children of different ages do to solve the dot problem: The youngest children (4–5 years) are unable to move beyond visual estimation. When it is suggested that they measure using rulers and straight edges, they either use them randomly or reject them altogether. They simply do not see the problem as one in which measurement is either possible or helpful—although some are glad enough to oblige the experimenter by appearing to measure if that is what is requested. Further, during this period of development, even visual estimation is very poor. Typically children can place the dot at the same height as in the other rectangle but are unable to take horizontal position into account at the same time, as shown in Fig. 7.4b. Within a few years, children's visual estimation improves, and they can place the dot in approximately the right position in both dimensions. But when asked to measure, children of 6 or 7 years seem unable to take both dimensions into account at once. They measure height and forget width, or vice versa. When they are estimating visually, they can coordinate height and width; but they are unable to coordinate the two dimensions in a rigorously quantified system (i.e., one that involves measurement).

In time, most children overcome this problem, but their first efforts are of a special kind. What they do is to lay the ruler along the line from the corner of the rectangle to the point in the first figure, as shown in Fig. 7.4c. Then they move the ruler over to the blank rectangle, trying all the while to maintain the slope of the ruler. Of course this introduces new kinds of inaccuracies, but at least it is an attempt to take both dimensions into account, even if it means reducing them to a single one. Only later, toward the age of 8 or 9 years, do most of Piaget's subjects recognize that to obtain a truly accurate placement of the second dot they need to make two separate measurements and them combine them. The subjects described by Piaget do this by measuring height and width separately and placing the dot at the intersection, as shown in Fig. 7.4d.

With this example, we can expand further our understanding of structure as conceived by Piaget. Here again, as with the triangles, we can see that accurate performance on even a simple geometric task involves understanding how several separate parts relate to each other. To place the dot correctly the child must act with respect to the whole, not just one dimension or the other. Further, using

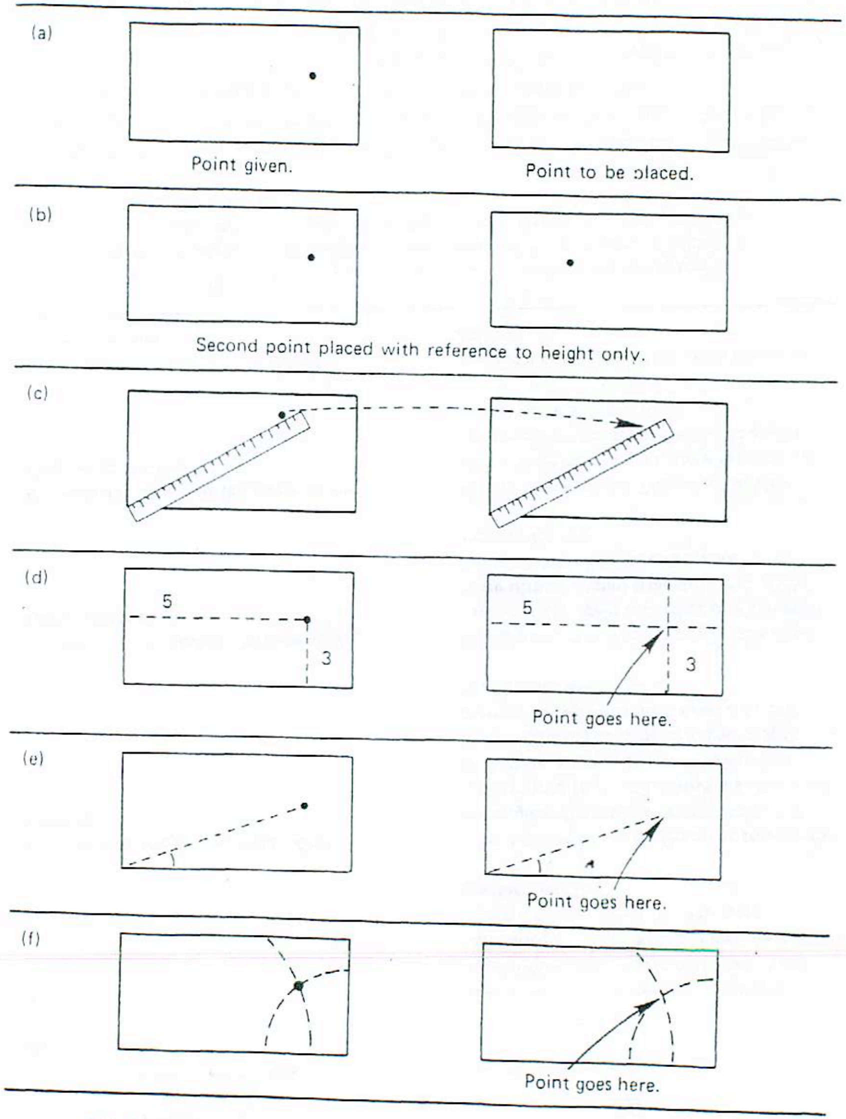


FIG. 7.4 The dot-in-a-rectangle problem. As children develop, they find increasingly sophisticated ways of figuring out where to place the dot in the empty rectangle.

rectangular coordinates, as opposed to simply estimating visually, requires imagining lines in an empty space and then partitioning those lines into equal units (the essence of measurement). Finally, with analysis and partitioning completed, all the actions must be coordinated so that the dot can be positioned. This problem illustrates the point that structure is not provided from the outside; children do not just look at the rectangle and know immediately where to place the dot. They have to engage in some intellectual activity to untangle the various aspects of the problem. In the present case the activity involves mentally constructing and using a rectangular coordinate system, which is the basis for much graphing and thus important not only in geometry but also in algebra, calculus, and other branches of mathematics. Children who accurately place the dot by this method behave as if there were a grid of horizontal and vertical lines, equally spaced, laid over the rectangle. They count up and across the imaginary grid to match the dot's position in one rectangle with a point in the empty rectangle. The grid is not physically there; the children must construct it. They can respond only by actively operating upon the material presented to them.

Piaget does not mention it in this context, but there are at least two other ways of accurately placing the dot. One method is to measure the distance from the dot to one corner of the rectangle and also to measure the angle created by the base of the rectangle and the line of measurement (Fig. 7.4e). These two measurements repeated on the empty rectangle will accurately determine the position of the dot. Another method is to measure the distance from the dot to each of two corners of the rectangle, construct arcs, and place the dot at the intersection of the arcs (Fig. 7.4f). Both of these methods require constructions by the child and require coordination of two separate measurements. They would thus illustrate Piaget's point about active structuring as well as the rectangular-coordinate-system method does. Yet Piaget's experiments in which rulers were provided but no protractors or compasses, allowed only the coordinate-system method to emerge. Because constraints of this kind are typical in Piaget's experiments, it is important not to take any particular study as evidence of a universal strategy of thought.

To summarize, actions—either mental (for it is not necessary *actually* to draw in all the lines) or physical (because they *could* be drawn in)—appear necessary for thinking and are part of Piaget's definition of structure. It is the nature of thinking to operate on the material presented rather than just accepting it. For Piaget, learning as well as performing mathematics is a matter of active thinking and of operating on the environment, not of passively noting or even memorizing what is presented.

With this characterization in mind it is useful to distinguish between gestalt and Piagetian definitions of structure, although the definitions are not mutually exclusive. The Piagetian examples used so far are from geometry, as were the problems of Wertheimer and Polya in Chapter 6. In both sets of examples,

"structure" refers to some representation of the subject matter or problem situation involving the relations of parts to the whole. In the gestalt sense, however, structure is something that is perceived because of a tendency to recognize particular organized wholes, or "good gestalts," in the environment. Piagetian structure, on the other hand, is something actively constructed by the human organism. The understanding that emerges from this activity bears a direct relation to what one might call the objective subject-matter structure, but it is really a personal creation, with consequent variations and fluctuations with time. As Piaget (1970) has put it, gestalt psychology deals with a structur*ed* system; Piagetian psychology deals with a structur*ing* system.

THE DEVELOPMENT OF PIAGETIAN STRUCTURES

Piaget's structuring system is dynamic, flexible, and capable of changing over time. Not surprisingly, then, his conception of structure is tied to a developmental theory of human intellect. It is to this theory we now turn, keeping in mind the Piagetian notion of active structuring as it is brought to bear on the performance of intellectual tasks.

The protocols from Piaget's experiments show that children become increasingly more sophisticated in their thinking as they become older. According to Piaget's interpretation, they take more characteristics of any situation into account and recognize how transformations in one part of an organized system will affect others. They are also able to carry out several operations, to recombine them, and to reverse their own operations mentally. This correlation of age with increasingly sophisticated thinking is central to Piaget's theory of intelligence and mental development. The essence of this theory is that, as people grow older, they do not just acquire more knowledge, they develop new, more complex cognitive structures.

Central to Piaget's developmental distinctions, as they relate to mathematical and other thinking, is the concept of an *operation*. We saw in the dot-in-a-rectangle task that successful solution required dealing actively with the materials presented. An operation is a special kind of mental action, special in that it can be reversed. It can be undone by performing another action. For example, a set of five blocks can be transformed by removing three blocks, but those blocks may be restored mentally if one wishes to deal with the original quantity. Thus, an operation can transform a system (subtracting 3 from 5 leaves 2), but another operation will restore the system to its original state (adding 3 to 2 yields 5). This possibility of making and undoing transformations, also referred to as *reversibility*, is a characteristic of structures of operational thinking. But it takes some time for operations, and operational thinking, to develop. In Piaget's theory the pres-

ence or absence of certain operations is a defining feature of the stages of development through which the individual passes on the way to intellectual maturity.

Quite young children, according to Piaget's research, do not think operationally at all. They can act upon things in the environment, but once an action has been performed, they are unable to keep in mind the way things looked before. Thus, they cannot mentally reverse their actions; in the technical language of Piaget, they have not yet achieved reversibility. Instead, Piaget characterizes children during this early stage of intellectual development as heavily influenced by the sensory and perceptual features of the events that surround them. The way things are presented to them are accepted as the way things are. Mental transformations, so characteristic of older children's and adult's thinking, cannot be performed, allowing perceptual givens to dominate thinking to a much greater degree.

This predominance of perceptual ways of thinking and the inability to think reversibly are illustrated by Piaget in his well-known studies of conservation and classification. In a number conservation experiment (Piaget, 1941/1952, p. 49), a boy is shown a row of flowers and a row of vases, lined up in one-to-one correspondence (see Fig.7.5a). In this arrangement the boy can see easily that there are the same number of flowers as there are vases. But next, the row of flowers is spread out, as the child watches. In this arrangement the flowers are no longer visually matched one for one to the vases, but none have been added or taken away (Fig. 7.5b). Despite having witnessed the transformation, and in some versions of the experiment actually performing it himself, the very young child no longer views the two sets of objects as being equal in number. According to Piaget, this is because once the flowers are spread out, and the one-to-one match destroyed, he cannot imagine them back in the original position. If someone puts them back, or asks him to, there is no problem. He will then state that the sets are equal again, but he will not see—despite repeated spreading and rematching—that the quantities do not change just because the spatial arrangement is transformed. Similar difficulties are encountered with conserving other kinds of quantities. For example, young children believe that the amount of liquid changes when water is poured from a low, flattish container to a tall narrow one, and they believe that the amount of clay changes when a ball of clay is rolled out into a snake-like form.

In the typical classification experiment (Piaget, 1941/1952, p. 165), a young girl is shown a collection of two white beads and seven brown ones, all made of wood. She agrees they are all wooden, and she may even count the full set. She compares the quantity of brown and white beads and easily determines that there are more brown ones. But next the experimenter asks whether there are more brown or more wooden beads. The girl's answer: more brown ones. According to Piaget, the perceptual dominance of the large number of brown beads interferes

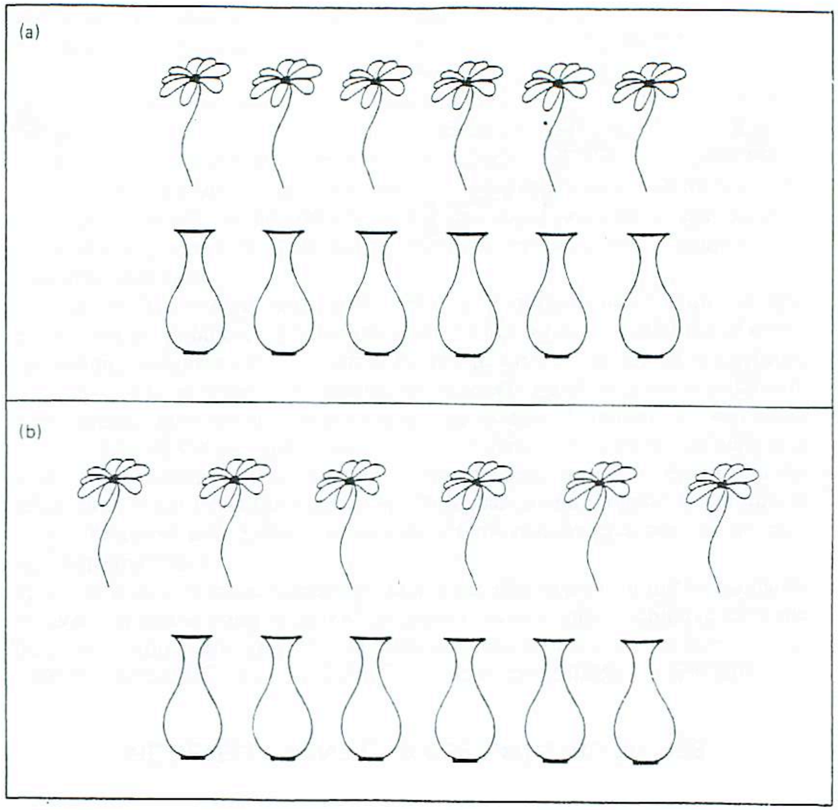


FIG. 7.5 A number conservation task. When the one-to-one match is destroyed (b), very young children judge that there are more flowers than vases because the row of flowers looks longer.

with the child's ability to take into account the fact that all of them are wooden. She compares brown with white again rather than brown with wooden. In other words, she seems incapable of comparing a subset with its own superset.

Dependence on perceptual features of objects or arrays and the inability to think reversibly are characteristic of what Piaget has labeled *preoperational* thinking. Preoperational thinking is presented in his work as typical of children of age 2 years up to about 6 or 7 years. After that, at just about the age when children are well launched in school, they enter what Piaget has termed the state of *concrete operations*. At this stage, according to Piaget, the children can think operationally: They can imagine undoing as well as doing transformations; they

can think in terms of more than one dimension at a time. And as they apply these enlarged capacities for logical reasoning to more and more ideas, their mathematical and scientific conceptions move closer and closer to those of adults.

The examples discussed earlier in this chapter were largely of children whose forms of thinking would be described as concrete operational. Recognizing the structural properties of triangles and using coordinate systems to organize space systematically both require operational thinking. Further, children capable of concrete operational thinking recognize easily and with conviction that changing the spatial arrangement of a set of objects does not change its number. They recognize, likewise, that the flowers and vases presented in the conservation task remain equal in number, that the amount of liquid stays the same when poured into a new container, and that there is the same weight of clay in the snake as in the ball. In the classification experiment such children are able to consider the color and material of the beads simultaneously while still recognizing them as independent features. As a result they are able to compare the superset of wooden beads with the subset of brown ones and to declare, as adults would, that there are obviously more wooden than brown beads.

In Piaget's theory, the achievement of the stage of concrete operations marks a turning point in children's intellectual development. Particularly in mathematics, far more sophisticated behaviors are possible with respect to quantity and spatial reasoning. In fact, many have argued that prior to the onset of operational thinking, any formal attempt to teach arithmetic and geometry will produce only limited understanding and limited ability to generalize and reason on one's own. But this is an argument that we can only assess in light of some of the major critiques and alternative interpretations of Piaget's findings that have been put forward in recent years. Before considering this critical literature, however, it is important to outline briefly the remainder of Piaget's stage theory.

The achievement of concrete operational thinking is great, but it is not the maximum that can be expected. According to Piaget, there is a stage of intellectual development beyond concrete operations, in which people are able to reason hypothetically and to take into account all logical possibilities. Called the period of *formal operations*, this stage typically develops with the onset of adolescence, and it involves the kind of thinking characteristic of the most advanced forms of mathematical and scientific reasoning. The following experiment exemplifies the changes that operational thinking is thought to undergo in Piagetian theory as an individual enters the stage of formal operations.

Separating Variables Experimentally: The Bending Rods Experiment

In this experiment (Inhelder & Piaget, 1958, p. 46), the child is given the task of predicting the conditions under which a rod will bend enough so that one end will touch the water in a basin. The rods vary in material (steel, brass, etc.), length, thickness, and cross-sectional form (round, square, rectangular). Three different weights can be screwed to the ends of the rods. The rods can be clamped to the edge of the water basin and can be shortened or lengthened according to where the clamp is tightened. The task is represented schematically in Fig. 7.6.

As in the Piagetian studies described earlier, the experimenter is interested not in whether the child already knows which characteristics affect the rods' flexibility but in how *the child goes about determining* the answers to such questions. The task requires considering each variable separately, and this means finding a way of holding all other variables constant while testing the effect of the one that is being considered. We begin with the case of a girl whom Inhelder and Piaget

THE BENDING RODS EXPERIMENT

Variables:

Metal: brass or steel

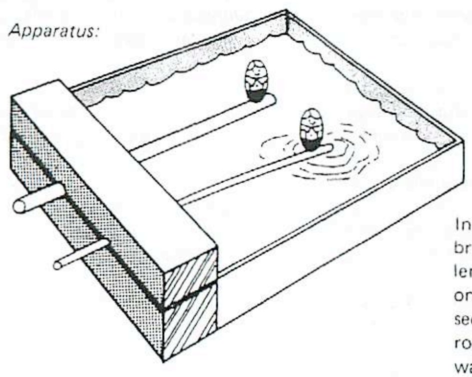
Length: adjustable by clamping at different points when attaching to side of basin

Weight attached to end of rod: 100 g, 200 g, or 300 g (weights shaped like dolls)

Cross-sectional form: round, square, or rectangular

Thickness of rod: thick or thin

Apparatus:



In this example--two brass rods, identical length, 200 g weights on both, same cross-sectional form--the thin rod bends to touch the water in the basin, but the thick one does not.

FIG. 7.6 A task requiring the ability to consider variables separately while holding all other variables constant. Children in the stage of formal operations can imagine the full range of possible combinations and construct experiments to test their hypotheses systematically.

presented as an example of a formal operational thinker. The protocol appears in Fig. 7.7.

Dei begins by listing the separate variables that might affect flexibility. She then takes up one variable after another and demonstrates its effect. She does this by allowing only one feature at a time to vary. Lest her perfect performance be attributable to chance, the experimenter checks Dei's understanding by asking her to use rods that vary in thickness to determine the effect of shape (d). Dei rejects this pair of rods and explains why.

Dei's performance is so confident and so logically perfect that it is almost hard to imagine people responding in any other way. But younger children have considerable difficulty and do not succeed in demonstrating the effects of each of

EXPERIMENTER	DEI
(a) Allows several experimental trials. *Tell me first what factors are at work here.*	*Weight, material, the length of the rod, perhaps the form.*
(b) *Can you prove your hypotheses?*	Compares the 200 g and 300 g weights on the same steel rod. *You see, the role of weight is demonstrated. For the materials I don't know.*
(c) *Take these steel ones and these copper ones.*	*I think I have to take two rods with the same form. Then to demonstrate the role of the metal I compare these two* (steel and brass, square, 50 cm long and 16 mm² cross section with 300 g on each) *or these two here* (steel and brass, square, round, 50 and 22 cm by 16 mm²). *For length I shorten that one* (50 cm brought down to 22). *To demonstrate the role of form, I can compare these two* (round brass and square brass, 50 cm and 16 mm² for each).
(d) *Can the same thing be proved with these two?* Indicates brass rods, round and square, 50 cm long and 16 and 7 mm² cross section.	*No, because that one* (7 mm²) *is much narrower.*
(e) *And the width?*	*I can compare these two* (round, brass, 50 cm long with 16 and 7 mm² cross section).

FIG. 7.7 Protocol from the bending rods experiment (Dei, 16 years, 10 months). (Adapted from Inhelder & Piaget [translated by Parsons & Milgram] 1958. Copyright 1958 by Basic Books, Inc. Reprinted by permission.)

EXPERIMENTER	BAU
(a)	Experiments with the rods. *Some of them bend more than others because they are lighter.* Points to the thinnest. *And the others are heavier.*
(b) *Show me that a light one can bend more than a heavy one.* Gives him a short rod, a long thin one, and a short thin one.	Places 200 g on the long thin rod and 200 g on the short thick one, without noticing the fact that the thin rod he has chosen is also the longest. *You see.*
(c) *Show me that the long one bends more than the short.*	Again puts 200 g on the same two rods. This time the result is supposed to demonstrate the role of length.
(d) *If I take away the long one, can you compare again to find out whether it's the lightest rod that bends more?*	*Yes, this one and that one.* Points to the two short rods, one thick, one thin.
(e) *Which is better, to compare these two or to compare the way you did before?*	*These two* (long and thin, short and thick).
(f) *Why?*	*They are more different.*

FIG. 7.8 Protocol from the bending rods experiment (Bau, 9 years, 2 months). (Adapted from Inhelder & Piaget [translated by Parsons & Milgram] 1958. Copyright 1958 by Basic Books, Inc. Reprinted by permission.)

the variables separately. Consider next a 9-year-old who has no conception of the "all other things being equal" notion in scientific proof (Fig. 7.8). Later we examine the responses of some children who are intermediate in their development.

Bau uses the terms *lighter* and *heavier* to refer to thinner and thicker rods respectively; the experimenter, who is interested in Bau's logical reasoning and not the terms he uses, simply adopts Bau's language. With this in mind, it is easy to see that Bau's problems are not simply linguistic. He is quite willing to draw conclusions about weight (thickness) when the rods being compared also differ in length. Conversely, he draws conclusions about length when the rods also differ in thickness (because he is using the very same pair of rods). Under pressure he is willing to follow the experimenter; thus, when only two rods are available, he will use them to compare for the effects of weight (d). Although we recognize this as the logically correct comparison, Bau's next response (e) makes it clear that he considers it a less valuable one. His final words clinch the case. He is looking not to test for the effects of the variables separately but to create situations as different as possible.

Bau is not totally unsystematic. He is able to observe and report what he sees accurately. Moreover he shows that he understands "multiplicative classification"—an achievement of concrete operational thinking—because he knows that rods varying in two dimensions are "more different" than rods that vary in only one (f). But as the number of variables in the problem increases, a more powerful logical system is needed, one that is able to construct mentally all the possible combinations, hypothesize effects of one or more of these combinations, and design appropriate demonstrations or experiments to test the hypotheses. These various steps require at least an intuitive understanding of propositional logic, a form of thinking that characterizes formal but not concrete operations.

Some intermediate cases (Fig. 7.9 and 7.10) now should complete the picture of how thinking changes between Bau's stage of development and Dei's. Dur nas the variables separate in his mind, but he does not quite figure out how to separate them spontaneously. His first attempt (Fig. 7.9b) uses rods that vary in both weight and thickness, although he does hold material and length constant. Pressed by the experimenter, he makes a correction that yields a properly controlled comparison (c). But we cannot see from this sample of his performance how certain he is of his method or whether he will perform logically on the next try. Kra is similar. He controls for weight and length in his first try (Fig. 7.10a and b) but allows material, shape, and thickness all to vary. Pressed by the experimenter for a rigorous proof, he is able to construct a good demonstration,

EXPERIMENTER	DUR
(a)	*There are flat ones, wider ones, and thinner ones, and longer ones. If they are both long and thin, they bend still more.*
(b) *Could you show me that a thin rod bends more than a wide one?*	Puts 100 g on the round steel rod (50 cm long and 16 mm² cross section) and 200 g on the round steel rod (50 cm long and 10 mm² cross section). *That one bends more* (10 mm² cross section).
(c) *I would like you to show me only that the thin one bends more than the wide. Is that way right?*	Takes off the 100 g weight and puts 200 g on the 16 mm² rod. *You see, this is the right way.*

FIG. 7.9 Protocol from the bending rods experiment (Dur, 11 years, 10 months). (Adapted from Inhelder & Piaget [translated by Parsons & Milgram] 1958. Copyright 1958 by Basic Books, Inc. Reprinted by permission).

EXPERIMENTER	KRA
(a) *Can you show me that a wide one bends less than the narrow?*	Puts 200 g on the round steel bar (50 cm long and 10 mm² cross-section) and 200 g on the square brass rod (50 cm long and 16 mm² cross section). *This one goes down more.* Points to the thin steel rod.
(b) *Why?*	*It is round, more flexible, the steel is less heavy, it is round and narrower.*
(c) *Fine, but I would like a rigorous proof that it's because it is narrower.*	Places 200 g on the round steel rod (50 cm long and 16 mm² cross section) and 200 g on the round steel rod (50 cm long and 10 mm² cross section). *You see, this one bends more because it is less wide.*
(d) *Bravo, can you demonstrate the same thing with others?*	*Yes.* Uses square steel rod (50 cm long and 16 mm² cross section) instead of round steel, so the comparison is no longer exact. *This one bends more, it is less heavy.* Points to the narrow round rod.
(e) *And can you demonstrate the role of the form?*	Puts 200 g on the rectangular brass rod (50 cm long and 16 mm² cross section).
(f) Points to round, steel rod. *Why does this one bend more?*	*Because it is round.*
(g) *Is that the only reason?*	*The brass is also heavier.* Spontaneously discards the steel rod and takes a square brass rod 50 cm long and 16 mm² cross section.

FIG. 7.10 Protocol from the bending rods experiment (Kra, 14 years, 1 month). (Adapted from Inhelder & Piaget [translated by Parsons & Milgram] 1958. Copyright 1958 by Basic Books, Inc. Reprinted by permission.)

allowing only thickness to vary (c). But he is not quite sure of his method. He next (d, e) tries allowing material as well as form to vary, but he finally corrects himself and controls for material (e).

For Inhelder and Piaget, both Dur and Kra represent borderline cases between concrete and formal modes of thinking. They have a nascent understanding of the need and method for separating variables, but they do not apply it systematically. These children's responses are considered valuable in pointing out the gradual

nature of the development of formal structures of thinking. Because they perform correctly once does not mean that they are well in command of the structures involved. It takes extended practice and experience for new logical structures to develop. Furthermore, these children's responses to the experimenter suggest that it is precisely during this transitional period that prompts and other forms of "instruction" have their greatest effect. This observation has special importance for education, and we return to its implications later in this chapter. First, however, we consider the general status of Piaget's stage and structure theories in light of recent research from an information-processing perspective.

CRITIQUES OF PIAGET

Perhaps inevitably, American psychologists—trained as experimentalists, initially uncomfortable with structuralist arguments and analyses, and predisposed to viewing the environment as a major determinant of human learning and development—have sought alternatives to Piaget's theory. Their critiques have been of three major types. One set of criticisms surrounds Piaget's stage theory 1 of cognitive development, questioning whether the idea of discrete stages, based on the emergence of certain logical structures, can withstand the test of closer analysis. A related line of criticism concerns the psychological reality of the logical structures at the heart of Piaget's theories, especially the—to experimen- 2 talists' eyes—rather loose linkage of the data to his conclusions about children's competence. Finally, many critiques are concerned with the claim, implicit in Piaget's writings, that children are biologically programmed for optimal de- 3 velopment in their normal social environment and that instruction can do little either to accelerate or to enhance the quality of children's logical functioning.

The Problem with Piagetian Stages

Most reports on Piaget's work, especially those intended to inform people concerned with teaching and instruction, place primary emphasis on the sequence of developmental stages: sensorimotor, preoperational, concrete operational, formal operational. These stages are often presented as if they were discrete time periods in children's lives and as if they set clear limits on the type of thinking that could be expected during each period. However, an extensive analysis of stage theory and related data by Flavell (1971) makes it clear that the stages are not all-or-none affairs; that is, children can behave as if they think reversibly and therefore operationally on one task, such as conservation of number, and still appear to be preoperational on some other closely related task, such as conservation of weight. Piaget acknowledges this phenomenon and has given it a name, *horizontal décalage*. But naming the phenomenon does not explain why it occurs, and the fact that it occurs represents a fairly serious challenge to a strict stage theory.

According to Piaget's stage theory, the acquisition of certain basic logical operations or structures such as reversibility allows children to recognize that spatial transformations (without adding or subtracting any quantity) leave quantities unchanged. But if the logical operations are present that allow a child to recognize conservation of number, why do they not also make it possible to recognize conservation of weight? Clearly other factors besides logical structures must be at work—factors such as the amount of experience a child has had with a particular quantity or material or the extent to which the critical features are easily visible or available for direct measurement. One can argue, for example, that conserving number or liquid quantity comes early because children have extensive experience—even before school begins—with counting and with pouring things from one container into another. One might also argue that in the young child's life there are more occasions requiring attention to a numerical or measured liquid quantity than occasions requiring attention to a weight. For example, when children share candy or drinks they do so on the basis of equal numbers or equal quantities. Rarely do they have to share on the basis of weight, nor are they often asked to compare weights with any accuracy.

None of these interpretations is necessarily contrary to Piaget's fundamental theory of structure in thinking, for Piaget would agree that the ability to deal logically with phonemena in the world is a result of the child's informal interactions with the normal environment. But these arguments do weaken the notion that logical structures, once attained, are so pervasive as to cause a distinct qualitative shift in the child's thinking. Instead of trying to pinpoint a stage of concrete operations, it seems more sensible to think of logical operations as one of many kinds of thinking abilities children acquire as they develop. This view suggests that a logical structure must be applied and "practiced" in a particular domain of knowledge before operational thinking in that domain becomes possible. For mathematics the implication is that experience with mathematical tasks, as opposed to more generalized experience with the environment, is more likely to improve children's ability to apply logical structures to their dealings with number, space, class inclusion, and the like.

The Difficulty of Assessing Competence

The notion that competence on specific tasks is a function of the growth of many skills is echoed in critiques that are not directly concerned with Piaget's stage theory. Some critics of Piaget raise the question: What does failure on a particular experimental task allow us to conclude about what children do and do not know? This question has often been formulated in terms of a distinction between

competence and *performance*. (See, for example, the discussions in Brainerd, 1978.) Piaget's interest is in the child's underlying competence, that is, in the logical structures that the child commands. But we cannot directly observe these logical structures. Rather, they must be inferred on the basis of the child's performance on particular tasks. Does failure to perform a particular version of a problem reflect a lack of competence with respect to the presumed underlying logical structures? Or can the failure be attributed instead to the salience of certain misleading cues that do not clearly communicate to the child what question is being asked? An article by Trabasso, Isen, Dolecki, McLanahan, Riley, and Tucker (1978), reviewing evidence from their own studies and those of other investigators, illustrates this line of criticism. The review shows that variations in presentation of the class inclusion task can make large differences in children's likelihood of recognizing that a superset has more items than any one subset.

Trabasso et al. (1978) argued that a child might understand hierarchical relationships in general (e.g., know that apples and pears are subsets of the superset fruits) but fail to recognize the applicability of hierarchical relationships to a particular display in an experiment. This would produce failure on the standard class inclusion problem. The researchers devised a number of experiments attempting to influence the likelihood that children would encode (i.e., notice or recognize) the hierarchical relationships. Simply having children label the sets ("These are cats. These are dogs. These are all pets.") was not enough to improve performance, but other manipulations did have an effect. For example, if the items used as subsets were considered *typical* of the superset in question (horses are typical of the superset animals, but flies are not), children were more likely to succeed on class inclusion. It also helped if there were two different supersets present (for example, fruits, with subsets of apples and pears, and vehicles, with subsets of trucks and cars). It especially helped to ask questions that demanded comparisons across the supersets ("Are there more pears or more vehicles?") as well as within them ("Are there more pears or more fruits?"). When the two different supersets were present, children answered one question about as well as the other, and the pear-fruit question was more likely to be answered correctly than if the vehicles were not present.

Other experiments showed that part of children's difficulty with the standard Piagetian form of the class inclusion task might be in understanding the question. A study was described in which black cows and white cows were the subsets. In the standard condition the children were asked, "Are there more black cows or more cows?" It was hypothesized that the absence of an adjective for the superset might invite young children implicitly to insert one. Since the only available contrast adjective was "white," they would transform the question to "Are there more black cows or more white cows?" and make the standard class inclusion "error." If the experiment was run with the cows either lying down or standing up, however, the experimenter could ask, "Are there more black cows or *standing* cows?" Adding this adjective greatly increased the proportion of correct responses. Other features, such as the size ratio of the subsets or the number of subsets shown for each superset, also influenced the probability of responding correctly.

All this variability argues convincingly that a child's failure on a standard Piagetian class inclusion test cannot automatically be taken to mean that the child has not developed a concept of class inclusion. Correct performance on a class inclusion task could depend on a number of separate variables—knowledge of which objects belong in which superset classes, the tendency to respond to linguistic constructions in adult-like ways (for example, without adding extra adjectives), the tendency to analyze displays hierarchically, etc. All of these factors must be taken into account, it is argued, to understand cognitive development, and the various abilities are not likely to develop simultaneously. Thus, the finding that a child has trouble comparing wooden beads with brown ones (the class inclusion task described earlier) should not be taken as evidence that the child has no concept of class inclusion. Instead the child may be missing any one of several different abilities or understandings that it takes to perform this particular version of the task. This does not mean, of course, that logical structures play no role in developing abilities or that they do not develop in a clear sequence. It does suggest that the status of certain logical structures that Piaget views as of primary importance in developmental growth may be less scientifically firm than many believe. And it certainly suggests that to design instruction based exclusively on Piagetian theory and analysis is to miss many important aspects of a good developmental theory of instruction.

Instruction on Piagetian Tasks

The interpretation of performance on class inclusion in terms of an interacting set of variables including but not limited to the use of certain logical structures is generally supported by a fairly extensive literature on attempts to teach children to perform the Piagetian tasks. This training research has had several motivations. Some American psychologists coming out of a behaviorist tradition were disturbed at the apparent implications of Piaget's theory for education. The theory seemed to imply that instruction could be effective only *after* structural changes associated with a given stage of development had occurred and, further, that these structural changes would emerge only as a result of very general kinds of experience—in other words, explicit instruction could make little difference. Some of the training research was aimed at disproving this aspect of Piaget's theory and showing that direct instruction could make a difference in tasks that were accepted as developmental landmarks. However, the psychologists who conducted most of the training studies were not interested primarily in the educational implications of Piaget's work. Rather, they were concerned with the mechanisms by which the environment brought about changes in cognitive com-

petence and with the specific things children had to learn in order to become operational thinkers. To analyze the roots of operational thinking in terms of component skills and knowledge instead of focusing on the holistic characteristics of thinking seems a very un-Piagetian strategy. Indeed this was the reaction of Piaget and his colleagues to most of the early instructional experimentation. It was expressed quite pointedly in a lecture Piaget gave in New York in 1967, which was quoted by Elkind (1970):

> If we accept the fact that there are stages of development, another question arises which I call "the American question," and I am asked it every time I come here. If there are stages that children reach at given norms of ages can we accelerate the stages? Do we have to go through each one of these stages, or can't we speed it up a bit? . . . It is probably possible to accelerate, but maximal acceleration is not desirable. There seems to be an optimal time. What this optimal time is will surely depend upon each individual and on the subject matter. We still need a great deal of research to know what the optimal time would be [p. 24].

Despite Piaget's disinterest, the American attempts at teaching Piagetian tasks yielded a great deal of information about how thinking abilities develop. They are also an interesting example of how psychologists working in one tradition can be enriched by contributions coming from quite a different tradition. In this case, the influence has gone both ways. Piaget has stimulated research on aspects of thinking that had not been considered before by American psychologists. At the same time, it is in large part because of American insistence on the importance of understanding the details of cognitive development that scholars from Geneva, working in Piaget's own research center, have turned to the problems of instruction (e.g., Sinclair, 1973). Let us consider some examples of these instructional experiments.

Gelman (1969) reasoned that in conservation of number tasks, and in other conservation tasks as well, the children acted as nonconservers (preoperational thinkers, in Piaget's terms) because they did not pay attention to the right features of the situation. In particular, they failed to recognize that there were several quantity dimensions, each of which had to be considered independently and yet simultaneously. Consider liquid quantity as an example. In a typical test for conservation of liquid quantity, the child is shown two glasses and asked to verify that they have the same amount of water in them. If the glasses are identical in shape and size, even very young children have no difficulty with this. But then the liquid from one of the glasses is poured into another glass that is, say, taller and thinner. Up to a certain point in their development, children typically say that the tall thin glass has more liquid, even though they have watched the pouring operation and know that nothing has been added or taken away. In this situation, there are several quantity attributes that the children could be attending to: overall size of the container, height of the liquid, width, etc. Yet

nonconservers appear to attend only to one of these, height. Could training be devised that would help children direct their attention to the most important of the attributes and to consider two or more of them simultaneously (i.e., separate the attributes and combine them appropriately)? This was Gelman's experimental problem.

The technique Gelman chose was to give children extended practice in making "same" and "different" judgments according to a named quantity dimension.

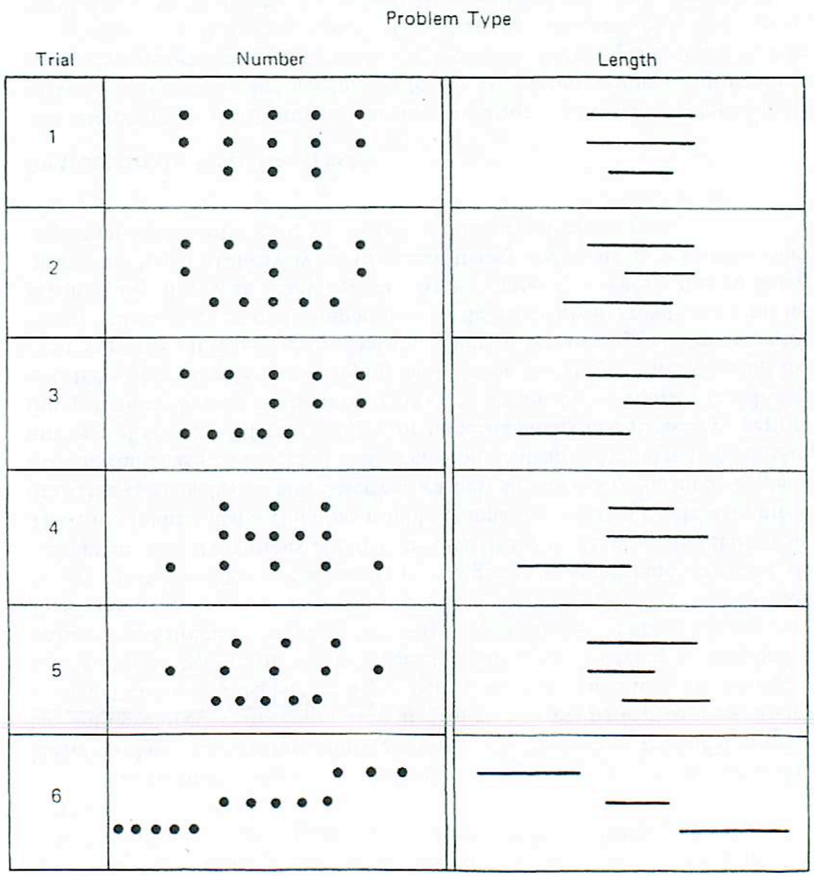


FIG. 7.11 Varied arrangements of stimuli (chips or sticks) used for practice in training conservation of number and length. (From Gelman, 1969. Copyright 1969 by the American Psychological Association. Reprinted by permission.)

Problems involving number and length were interspersed to show the children that a dimension that was relevant one time might not be relevant the next; that is, they must always consider the question being asked as well as the way the display looks. Figure 7.11 shows the kinds of stimulus arrangements that were used. For a number problem the experimenter would say, "Show me two rows that have the same number of things," or, "Show me two rows that have a different number of things." For length problems the task was, "Show me two sticks that are the same (or different) in length." After each response, children were told whether they were correct and were given trinkets as prizes for correct answers. After extended practice (about 100 trials each for number and length) virtually all the kindergarten children in the experiment gave the right responses 95% of the time. Thus, they had successfully learned to discriminate different quantity dimensions. Did this mean they would perform better on conservation tests than they had at the beginning of the experiment?

On the length test 18 of the 20 trained children scored perfectly, and on the number conservation test 19 scored perfectly. This contrasted with much lower scores for children who had not been told whether they were correct during training or who had been trained on a picture-matching task that did not involve quantity discriminations. Most of the trained children continued to perform perfectly 3 weeks later on a retention test. Gelman also tested the children on other conservation tasks—liquid quantity and mass—for which they had not been trained. Among the trained children, there was considerable transfer of the discrimination skill to these very different kinds of tasks. Although this transfer was far from perfect, it nevertheless indicated that children had acquired the habit of examining quantities carefully and that this could transfer to other dimensions as well. As in the class inclusion experiments, then, we have evidence that if children attend to or encode the relevant aspects of the stimuli, they can solve a problem that Piaget would have concluded was beyond their logical capacities. But this study, unlike the ones discussed by Trabasso et al. (1978), suggests that *some* fundamental principle was acquired, even if it was not the concrete logical operations claimed by Piaget. The tendency to conserve after training was carried over to dimensions other than number and length, so the children must have learned not only which dimensions to attend to but also the general concept that physical transformations are irrelevant to quantity.

A study by Bearison (1969) showed even wider transfer across different conservation tasks and demonstrated retention of the effect for 6 months—long enough for us to infer that it was probably a permanent concept for the children and not just a short-term effect such as is common in many training experiments. Bearison reasoned that if children could be made to understand the principle of measurement (i.e., that a single unit is applied repeatedly to a quantity), then they would be freed from the influence of perceptual features in the immediate displays when making quantity comparisons. They would understand that if two displays had the same number of units, then the quantities were equal regardless of how the units might be arranged or presented.

Bearison used the liquid quantity problem as the basis for his experiment. In the initial phase of training, white-colored liquid (called milk) was poured into many small 30-milliliter beakers—the kind that are sometimes used for dosing out medicines—and various numbers of these beakers were assigned to the experimenter and to the child. On each trial, the child had to decide who had the most milk on the basis of the number of filled beakers each person had. He was thus taught that quantity, even a "continuous" quantity like fluid, could be compared on the basis of a measured number. In the second phase of training, either the child's or the experimenter's liquid was transferred to a single larger beaker. The child did the pouring, counting the beakers as he poured in each one. Then, with the empty beakers still present so that they could be counted, the child had to compare the quantity of the experimenter's liquid with his own. There were numerous trials, in which the experimenter and child sometimes had the same number of beakers and sometimes an unequal number. Before passing on to the next phase, children had to demonstrate not only that they could decide correctly who had the most milk but also that they could justify the decision in terms of the number of separate small beakers involved. In the third phase of training, both the experimenter's and the child's beakers were transferred into larger containers, and comparisons were made as in the previous stage.

In the fourth phase the small beakers were not used; rather, milk was assigned to the experimenter and the child in two large, identical containers. Milk from one of the identical containers was then poured into another container of a different shape. It should be apparent that phase 4 amounts to Piaget's own conservation test, and quite a conceptual leap is required. In the experiment, children were permitted to go back and forth between the phase 3 training and the phase 4 conservation test as many times as needed to help them see the connection between the two situations.

Not surprisingly, the children who received Bearison's training learned to conserve liquid quantity, whereas untrained children did not. More interesting is the fact that the trained children learned to conserve in various other dimensions as well. The percentage of conservers was highest for "discontinuous" (discrete) quantity (same training procedure but using beads instead of liquid in the containers). Number, length, and mass (amount of clay) were also conserved by a significant number of the trained children, as compared with a control group. The 7-month posttest yielded the most persuasive data of all. There was some improvement in the number of conservers in all areas among both experimental (trained) and control children. But the trained children maintained their advantage in every area tested, including two—continuous and discontinuous *area*—in which they had not shown any advantage on the earlier test! Here we see strong evidence not only of retention but also of a kind of generalization that suggests a

very fundamental principle was acquired and, over time, applied to more and more situations by more and more of the children. About three-quarters of the children were conservers 7 months after the training, even though they were only kindergarteners.

A Neo-Piagetian Analysis

We spent a good deal of time describing the Gelman and Bearison experiments because they are examples of successful instructional efforts in an area that for a long time had known few successes. The pages of developmental psychology journals are filled with reported attempts to teach children Piagetian tasks, but many of these studies failed or had very limited success: They showed only short-term effects and little or no transfer to related tasks. What do the successful training studies have in common, and what seems to account for their success? Many answers to this question have been offered, but few psychologists have attempted to analyze Piaget's theory specifically in terms of its implications for instruction. An exception is found in the work of Case (1978), who has proposed both a sequence of analytic steps and a set of variables with which to analyze Piagetian tasks.

Calling himself a "neo-Piagetian," Case explicitly accepts the fundamental features of Piaget's theory of cognitive development. He argues, however, that the difficulty of extrapolating from Piagetian theory to instruction arises because the theory is *structural* rather than *functional*. Piaget's protocols of children performing tasks offer us still snapshots of cognitive structures at various levels of functioning. They do not offer the kind of detailed psychological model of the acquisition and development of structures that is needed for designing instruction. In other words, Case calls for process models of Piagetian structures and of the transitions they undergo during specific learning episodes. In order to specify the nature of such transitions, one needs to analyze the mental steps required in cognitive tasks and to assess children's current performance on those tasks at various levels of development. This corresponds to the rational process analysis and empirical analysis that we describe in Chapters 3 and 4. Detailed work of this kind on Piagetian tasks has been carried out by information-processing psychologists: Klahr and Wallace (1972, 1973) have simulated conservation and class inclusion behavior on computers; Baylor and Gascon (1974) and Young (1973) have developed computer models that attempt to explain the behavior of children on seriation tasks, including their typical errors. None of these efforts, however, has directly addressed Case's concern, the analysis of Piagetian tasks for instructional purposes.

Hard to apply

Case argues that the difficulty children have with particular tasks is a function of (1) mental actions or *schemes*[1] they have available to them; (2) the number of schemes they are able to activate simultaneously; and (3) the strategy used in performing the task. All of these factors interact, of course, and the way they manifest themselves is a function of particular features of the task presentation. Other factors that influence task performance are the tendency of individuals to be influenced by the surrounding perceptual field ("field dependence" or "field independence"), the salience of relevant information in the problem situation, and the child's disposition or motivation to resolve the cognitive "conflict" (i.e., the discrepant information arising from an interpretation of the task features).

From a developmental point of view, the most important feature of Case's theory is the notion that as they grow older, children can activate an increasingly large number of schemes simultaneously. Building on prior work of Pascual-Leone (1970), Case defines a construct called the *M-space* (for *memory space*). This corresponds in a rough way to the concept of limited working-memory capacity discussed in Chapter 2, but the schemes that constitute "chunks" in M-space theory are larger than those in many working memory experiments. Different tasks require different numbers of schemes for successful solution, but complex tasks can sometimes be solved by using strategies that reduce the number of schemes needed simultaneously. Of course, discovering a simplifying strategy for oneself would require more M-space than just applying such a strategy. This means that if specific strategies are taught, they may make it possible for younger children (with smaller M-spaces) to perform tasks normally performed only by older people. Thus, specific knowledge affects one's performance level. This, Case argues, is why certain training studies work. Changing the task—for example, so that fewer competing schemes are activated by the visual display—can also help people with smaller M-spaces. This might account for why task modifications such as those described by Trabasso et al. (1978) for class inclusion could make such a difference.

Case's analysis of the liquid conservation task gives a sense of how these various factors interact. This is the task in which water from one of two equal beakers is poured into a tall, thin beaker and the child compares one of the original beakers (A) to the new, tall beaker (B). There are several ways of responding to this task. The following is one example:

1. Scan the vertical dimension of the water in Beaker A.
2. Scan the vertical dimension of the water in Beaker B (noting that the water in B continues past the point where the water in A ends).
3. Recognize that Beaker B contains a taller column of water.
4. Conclude that Beaker B contains more water than Beaker A [Case, 1978, p. 178].

[1] *Scheme* is a term used by Piaget to describe organized patterns of behavior (mental *and* physical) that occur in relation to the environment, a small-scale version of a structure. Case adopts the term in describing mental behavior at the level of specific tasks. In connection with a liquid conversation task, for example, Case refers to a *figurative* scheme (keeping in mind how the two original beakers looked); various *operative* schemes (scanning the two beakers; storing information in long-term memory); and an *executive* scheme (a control function that keeps in mind the series of steps for solving a problem).

This response is typical of very young children, who respond almost exclusively in terms of height. With respect to Case's theory the important point is that only a single scheme (the operation of comparing heights) must be attended to. If height could not be directly (i.e., visually) compared, more M-space would be needed, because two heights would have to be separately noted and then compared. Presumably, Case would also appeal to the M-space construct to account for the greater ease of number conservation when the sets are small enough to be subitized in parallel: Less M-space is required.

Another way of dealing with the liquid conservation task is to scan both dimensions, but without considering them simultaneously:

1. Scan the vertical dimension of the water in Beaker A.
2. Scan the vertical dimension of the water in Beaker B (noting that the water in B continues past the point where the water in A ends).
3. Recognize that Beaker B contains a taller column of water.
4. If the difference is large, conclude that B contains more water than A; otherwise, proceed to step 5.
5. Scan the horizontal dimension of the water in Beaker B.
6. Scan the horizontal dimension of the water in Beaker A.
7. Note that Beaker A contains a wider column of liquid.
8. If the difference is large, conclude that Beaker A contains more water than B. Otherwise, recycle to step 1, setting the criterion for "large" at a lower value [Case, 1978, pp. 179–180].

This strategy requires two simultaneous schemes, because the width comparison cannot be made on a direct visual basis. Further, if height and width differences were to be compared, three schemes would be needed, because height would be stored while the width comparison proceeded. A strategy of this kind would be required for noticing the "conflict" between height and width: Beaker A contains a wider column of liquid but Beaker B contains a taller column—they cannot *both* hold the larger quantity of liquid. In Piaget's and in Case's theories, such cognitive conflict is assumed to be the the incentive behind the mental work that leads to transitions to new levels of development; children work to resolve the conflict. In the ordinary environment, then, a relatively well-developed M-space that could accommodate up to three schemes (typical at about 7–9 years of age according to Pascual-Leone, 1970) would be needed before a child could spontaneously develop liquid conservation. Extrapolating to instruction, we might suggest training that would draw attention to the conflict by highlighting specific features of the stimuli (this is what Gelman's [1969] training did, for example) or by teaching specific strategies for making comparisons (as Bearison's [1969] did). But unless a strategy were taught that actually called on fewer schemes, Case's theory would predict no dramatic decrease in the age at which children could learn to perform specific tasks.

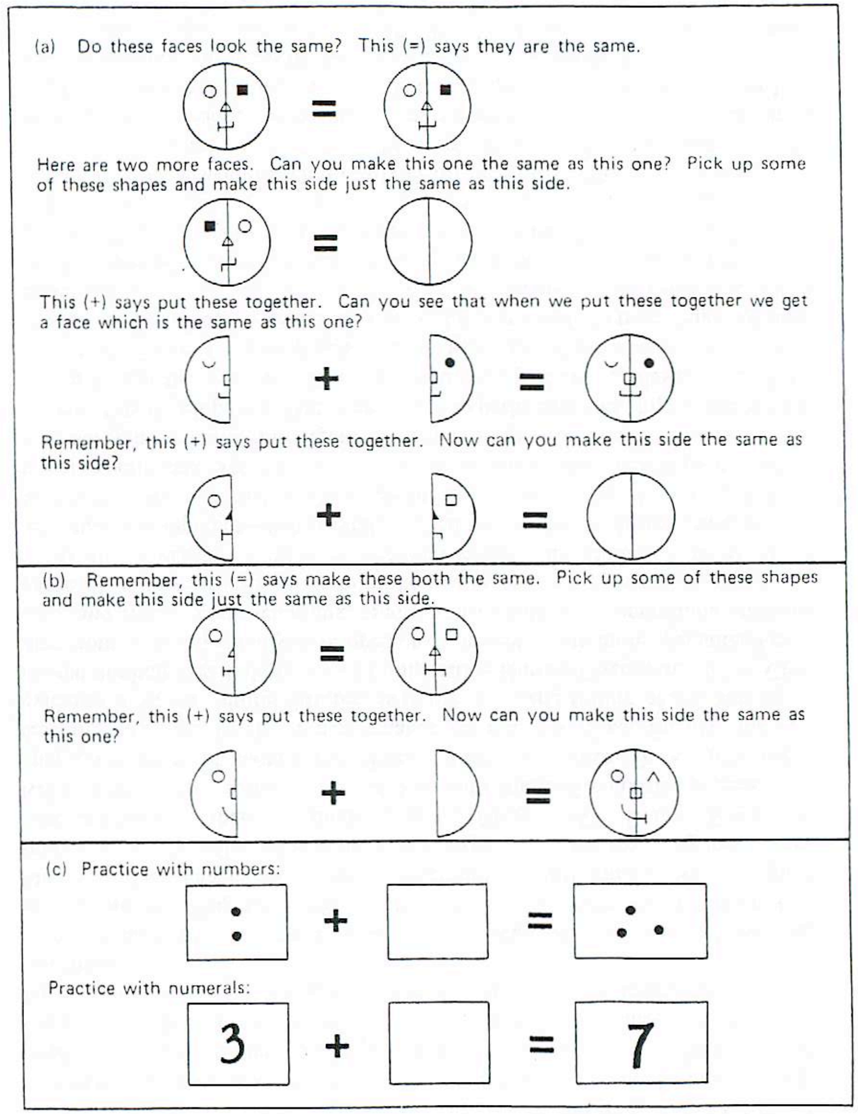


FIG. 7.12 A neo-Piagetian approach to the missing addend problem. (Adapted from Case, 1978. Copyright 1978 by Lawrence Erlbaum Associates, Inc. Reprinted by permission.)

Case has shown that tasks can sometimes be simplified dramatically by teaching the fundamental concept in a way that reduces the number of schemes that must be handled simultaneously and by teaching a relatively simple solution strategy. An example is the solution of missing addend problems, such as 4 + □ = 7, a task common in first-grade mathematics texts, but one that first-grade children find very difficult. Case worked with kindergarten children, presenting the series of exercises shown in Fig. 7.12. The first part of the procedure (a) familiarizes the student with the basic form of the missing addend problem. It uses only familiar elements (shapes, faces), and it maximizes the salience of cues (equation format, operational signs). It also reduces the number of schemes involved by eliminating the need to read numbers, to count, or to hold numbers in mind. The instructional emphasis is on matching both sides of the equation. The second part of the procedure (b) is designed to show the child that the common strategy of adding the first and last terms of the equation (4 + □ = 7; answer: 11) is counterproductive to the goal of matching both sides of the equation. Finally, the third part of the procedure (c) reintroduces the complexities originally stripped from the problem (such as numbers and counting). It provides reminders of the meaning and importance of the relevant cues (+ and = signs) and practice in executing the required task. Case also notes that young children who are successful in solving missing addend problems use the procedure of counting up from the given addend rather than subtracting it from the sum (cf. Groen & Poll, 1973). This requires fewer schemes than the adult strategy of subtracting 4 from 7.

PIAGET AND INSTRUCTION

It is no accident that our first discussion of instruction in this chapter came in the context of critiques and reinterpretations of Piaget's theory rather than in the presentation of the theory itself. In his own work, Piaget has not focused on questions of instruction. He considers himself a student of human knowledge and development, not of educational design. It falls to those of us who are concerned with instruction, therefore, to try to determine the implications of Piaget's theory for the teaching of mathematics. Many lessons have been drawn from Piaget and a surprising variety of specific educational practices and recommendations attributed to the influence of his theory. In the remainder of this chapter we consider some of these proposals, asking in what sense they are strictly Piagetian in origin, and how useful they seem as instructional guidelines.

Accelerating Progression Through the Piagetian Stages

Anyone who becomes familiar with the details of Piaget's research cannot help but be impressed by the ingenuity of the tasks used in his experiments. The tasks involve high levels of mathematical and logical thinking, in forms that appear capable of capturing and maintaining the interest of children. It might seem natural, therefore, to take the tasks as important in themselves and to set about developing curricula and teaching strategies that will accelerate the age at which children can perform them. To the extent that the tasks clearly reflect different stages of intellectual functioning, such a program would have the effect of accelerating progression through the stages of development described by Piagetian theory.

As we have seen, laboratory studies demonstrate the possibility of teaching various specific Piagetian tasks, with degrees of transfer and retention that suggest stable changes in cognitive functioning. Furthermore, whereas most studies show little effect of specific school experiences on acquiring more complex operational modes of thinking (e.g., Almy, 1970; Beilin, Kagan, & Rabinowitz, 1966), some studies in less industrialized countries suggest that appropriate forms of mathematics (Prince, 1968) or science (Duckworth, 1964) teaching can raise levels of performance on Piaget's tasks. Finally, there is evidence of broad cultural differences in the onset and timing of concrete operational thinking (see Laboratory of Comparative Human Cognition, 1979). Children from more industrially developed environments, with more specialization of labor and at least some schooling, tend to show evidence of operational thinking earlier than those from rural and unschooled areas.

All this suggests—quite in accord with Piaget's interactionist theory of intellectual development—that it might indeed be possible to design curricula and use teaching methods that would accelerate operational forms of thinking. Educational experiments exploring this possibility should greatly interest psychologists who study how human intelligence responds to environmental influence in the course of its development. But even if such experiments were highly successful, it is not clear that such acceleration would be a particularly desirable goal for education. There are also practical considerations, illustrated by the training studies we described. Whereas they proved it is possible to treat Piagetian tasks as subject matter—that is, to teach them as other mathematical content might be taught—they also demonstrated that doing so is likely to involve extended and complex forms of instruction. Consider that in both the Gelman and the Bearison experiments all instruction was done on a one-to-one basis, one child at a time. On a larger scale would such a costly effort be worthwhile?

The crucial question is whether, by accelerating development, we could improve people's chances of becoming, in the long run, better as well as earlier thinkers. Our current evidence suggests that people in all cultures sooner or later acquire the abilities necessary for at least concrete operational thinking. They all learn, without explicit instruction, to classify objects along more than one dimension, to conserve, to analyze geometrical figures in terms of interrelated structural features, and so forth. They do not have to be taught these things explicitly; they seem to acquire them through the normal transactions of living, even in quite "undeveloped" societies. Perhaps, then, we need to ask ourselves whether

it is worth spending instructional time on things that will almost certainly be learned anyway. If the answer is no, as we believe it is, then focusing mathematics instruction on the acquisition of concrete operations would be a mistake.

For formal operations, however, the case is not so clear. Formal operations have been much less widely studied than concrete operations. If anything, there is even less clarity about which tasks in which forms actually reflect the structures that Piaget attributes to formal operational thinking. Nevertheless, it is clear that there do exist complex modes of reasoning, often associated with scientific thinking, that are difficult to attain and to use. Several investigators (Laboratory of Comparative Human Cognition, 1979) have concluded that a few cultures never arrive at fully developed formal operational thinking of these kinds. More important for our purposes, some individuals do not learn formal modes of thinking even in the most intellectually advanced cultures. Here indeed, if this evidence is correct, is a challenge worth considering. Can educational programs be developed that foster the development of formal logical modes of thinking, so that more people have an opportunity to achieve the levels of intellectual functioning now achieved by only a relative few? Can more people learn scientific modes of reasoning based on logic and mathematics? What would programs with such goals look like?

Although studies addressing this question are scarce, there is evidence that certain kinds of instruction may make it possible for preadolescent children to solve problems requiring formal operations. Siegler and his colleagues (Siegler, & Liebert, 1975; Siegler, Liebert, & Liebert, 1973) have taught children to sort out the effects of different variables on Inhelder and Piaget's (1958) pendulum problem. In this experiment, children observe the effects of varying weights and lengths of string upon the motion of a pendulum. The speed of the swinging motion is an effect of the length of the string to which the pendulum weight is attached, the actual weight being irrelevant. Inhelder and Piaget's 10- to 12-year-olds tended to attribute the speed to the weight of the object attached to the string. Siegler provided his subjects with conceptual training (definitions of dimensions and levels of factors, and the applicability of those concepts to problem solution) and gave them analogous problems to solve. Building a conceptual framework for scientific thinking and exercising the new concepts in analog tasks allowed these 10- and 11-year-olds to solve the pendulum problem quite readily. Case (1978) reports an experiment in which children as young as 8 years learned to separate variables in the bending rods task we described earlier. The method involved highlighting the separate variables (for example, by actually weighing rods and blocks) and providing a "stripped-down" version of the task for early practice. Results like Siegler's and Case's cast doubt on Inhelder and Piaget's implication that one must await children's readiness for formal operations training. At the moment, however, we have only a few experiments that hint at the possibilities for direct instruction in formal thinking. There is no

organized curriculum to point to as an example whose effectiveness might be studied and evaluated.

Matching Instruction to Developmental Stages

There is another, more general way in which Piaget's sequence and stages of cognitive development might guide mathematics teaching. If his theory of development is correct, at least in broad outline, then it seems to set limits on the kind of reasoning and understanding we can expect from children at any particular point in their development. This implies that both the content and presentational techniques of teaching should be matched to the child's current level of development.

What would "matching" mean in instruction? How can the teacher respect children's current understanding and forms of thinking and at the same time play a significant role in their development? On first thought it appears that a curriculum designed to match children's developmental level should not expect them to engage in activities they are not yet fully able to understand. Thus, for example, one should not teach anything about addition until basic number concepts are well established and until laws such as commutativity and associativity are understood. Nor should one teach measurement until conservation of length is established. This might be called a "readiness" conception of how to match instruction to development. It assumes that fundamental understanding comes about through maturational processes or through some generalized environmental exposure.

Although this conception of matching has a distinguished history in psychology, it is virtually a counsel of despair for education, because it implies there is little for teachers to do except to await certain developments. If development is delayed or incomplete in some children, little is suggested in the way of educational activity to remedy the situation. Furthermore, the readiness conception gives little credence to the power of the environment, including the instructional environment, to influence the course of development.

A more positive definition of matching is available and has a growing basis in empirical experiment. It recognizes the ability, indeed the responsibility, of the teacher to influence important aspects of children's intellectual development. According to this conception, most fully developed by Hunt (1961, 1969), the important thing in education is always to pose problems that are slightly beyond the learner's current capability but not so far beyond that they are incomprehensible. Hunt draws on Piaget's notion of cognitive conflict, which is produced partly by the internal workings of intelligence and is often prompted by normal transactions with the social and physical environment. Cognitive conflict is what eventually presses individuals toward new and more powerful ways of thinking. He suggests that certain forms of instruction—defined as organizing the envi-

ronment so that it makes new but possible demands—can foster structural reorganization and thus contribute both to the learning of specific new information and to overall cognitive development.

Lovell (1971) adopts a similar view:

> It is not in any sense suggested that the child must always be "ready" for a particular idea before the teacher introduces it. The job of the teacher is to use his professional skill and provide learning situations for the child which demand thinking skills just ahead of those which are available to him. It is a question of keeping the carrot just ahead of the donkey's nose. When a child is almost ready for an idea, the learning situation provided by the teacher may well "precipitate" the child's understanding of that idea [p. 17].

Lovell attempts to apply this principle directly to the problem of teaching mathematics during the primary school years. In keeping with his general understanding of Piaget's theory of intellectual development, the topics and tasks proposed by Lovell focus on the conceptual bases of mathematics and help children develop their understanding of these concepts; computational skill is not a central concern. His suggestions cover topics in number and set, operations on mathematical sentences, geometry and space, mappings, and pictorial representation. However, these topics and tasks are only loosely tied to Piaget's theory and research. Lovell acknowledges the difficulty of outlining Piagetian teaching implications for specific mathematical tasks: "Piaget's work cannot tell us which mathematical ideas should be introduced to children; only mathematicians and teachers can do that." The choice of topics and tasks, in other words, is defined by the subject matter of mathematics. But Lovell feels Piagetian research and theory can guide how they are taught by suggesting difficulties that children are likely to have at certain points in development and by providing a general framework of instructional principles.

For both Hunt and Lovell, then, the value of Piaget's work for matching instruction to learner capabilities lies not in any task he has devised or in the specific developmental stages he outlines. It lies in his general characterization of the quality of children's thinking and the instructional principles that can be derived from it. However, although the instructional principles proposed—such as constructive learning and the use of concrete representations for concepts—seem congruent with Piaget, they do not seem to derive uniquely from his theory. Indeed many of the "Piagetian" principles will be familiar from our description of structure-oriented teaching materials and methods in Chapter 5. It is nonetheless useful to review briefly several of these principles here, concentrating for the moment on their derivation from Piagetian theory.

Constructive Learning. "To understand," Piaget (1973) has said, "is to invent," to build for oneself. Although children can be helped to acquire

mathematical concepts by means of special materials and teachers' questions, it is only through their own efforts that they will truly understand. Constructive learning thus implies activity by the learner, but activity of a special kind. The "active responding" called for in behavioral approaches to teaching is designed to provide an occasion for reward; the expected responses are largely specified by the teacher. In contrast, the activity called for by Piaget centers on trying to develop one's own approaches to particular tasks and problems. It is activity in which errors may be frequent, but these errors are part of the child's attempt to make sense of concepts. Constructive learning involves "trying out" ideas, testing to see which solutions work and which do not. This requires learning materials and learning environments that provide feedback to the individual on the outcomes of these trials. For Piaget, the kind of feedback that helps in this constructive learning process includes information from both the physical and the social environment.

Concrete Representations. As Piaget's experiments demonstrate, young children are able to think operationally only with respect to actually present materials and situations. They require feedback from the physical environment, in the form of concrete representations of concepts. And yet our educational system often depends almost exclusively on verbalization of ideas, in both teaching and testing. According to Piaget, verbalization does not ensure understanding, nor does understanding depend on verbalization. This being the case, instruction in a purely verbal mode is bound to fail, particularly when new concepts, demanding reorganization of thought structures, are being taught. In contrast, pictorial and concrete representations of mathematical concepts provide direct feedback on the correctness of children's tentative understandings. Thus, the various structure-based teaching materials cited in Chapter 5 may be taken as examples of Piagetian principles, although they do not directly derive from Piaget's work nor use the tasks characteristic of Piaget's research.

The Social Environment for Learning. The social environment provides the second kind of feedback that prompts children to give up old conceptions and structures and build new ones. According to Piaget, the kind of structural reorganization that is integral to the process of intellectual development comes about, in part, when children encounter disbelief of their proposals. As their social world expands with increasing age, they discover that other people do not always agree with their view of reality. Presumably this discovery leads children to examine more closely their own beliefs, to engage in closer observations and tests of the physical environment, and eventually to revise their conceptual structures. In this process, Piaget suggests, the disagreement of adults is less influential than the disagreement of children who are close to them in age and general conceptual level. If this is the case, then children's learning depends to

an important degree on the social environment and the opportunity it provides to interact with peers over intellectual tasks.

To test this notion, Murray (1972) examined the effect of informal discussions among children upon the development of a basic concept, conservation. Murray pretested kindergarteners and first graders on six different conservation problems. He then organized the children into groups of three, with two conservers and one nonconserver in each group. Each group was required to arrive at collective answers to a series of conservation problems. There was no direct attempt to train the children in conservation; they were told only to discuss each problem, to explain to each other why they held the opinions they did, and eventually to agree upon an answer. A week later, every child was posttested on the conservation problems, this time alone. All the children improved their performance, the original nonconservers most of all. Because some of the posttest problems were quite different from the training problems, it is likely that at least some of this improvement could be attributed to social interaction and the idea testing induced by it.

The notion that social interaction plays a role in inducing the cognitive conflict that is the precursor of intellectual growth has often been cited in support of "open" or informal classroom environments in which extensive informal child-to-child interaction is possible. No good evaluations of the effects of such environments exist, however. In any case, their effectiveness would undoubtedly depend on the extent to which adequately structured and developmentally appropriate tasks are available for children to work on. Within the constraints of ordinary classrooms, providing structure without the straitjacket of simple drill exercises has so far proved an elusive, although still desirable, goal for educators in mathematics as in other fields.

Teaching as Clinical Interaction. Piaget's research is built on a special kind of interview technique, examples of which have appeared in the course of this chapter. The strategy is to set a distinct problem, embodied in physical objects that the child experiments with in the course of the interview. Both verbal responses and physical actions provide the data from which thought processes are inferred. The interview method itself, although not often thought of in this way, may constitute one of Piaget's greatest contributions to education. It provides a means by which teachers can understand what children understand. This is a crucial step in an educational strategy that seeks to match instruction to children's development. Teachers can cultivate their own skills of observing and questioning, as well as of setting interesting problems. As they become better at this, they begin to note details of children's thinking that had not been apparent before and find themselves able to follow children's lines of reasoning more clearly. Under these conditions, mistakes are not seen as poor thinking but as information about each child's current understanding. On this basis, tasks and questions can be posed that represent the best match in terms of the intellectual "stretching" propounded by Hunt (1961, 1969) and Lovell (1971).

This kind of clinical teaching could be applied in any substantive area, but it should not be thought of as independent of subject matter. Rather, it requires the kind of solid understanding of the subject matter that allows the teacher to recognize sensible but unusual responses and to invent problems that probe a child's understanding. Only teachers who themselves understand the conceptual bases of the mathematics they teach will be successful at the clinical technique. This understanding must come from studying mathematics, perhaps in ways specially adapted to questions of pedagogy. It will not come from studying Piaget or any other psychological theory that does not directly address mathematics learning and thinking.

SUMMARY

The developmental theory of Piaget emphasizes the dynamic aspect of intellectual activity and the psychological structures characteristic of children at different levels of development. In the writings of Piaget, the term *structure* is used as a means of describing the organization of experience by an active learner. Protocols of children engaged in mathematical and logical tasks are interpreted as evidence of qualitatively different cognitive structures that permit different understandings and solutions of the tasks. These differences in cognitive structures are said to develop in a sequence encompassing defined stages. During the normal period of schooling, children typically pass from the stage of preoperational thought through the stage of concrete operations to the stage of formal operations. Concrete operational thinking implies being able to reverse sequences of action mentally and to test various hypotheses. Formal operational thought implies an ability to think abstractly and to plan systematic variations in problem elements.

Critics of Piagetian theory question the reality of the stages, because there is so much variability in children's performances on tasks that presumably depend on the same operations. Noting the way in which task modifications can radically alter task difficulty, critics suggest that a number of variables, not only the logical structures that Piaget stresses, are required to account for performance; it is therefore not possible to infer a lack of logical competence on the basis of a particular performance. Neo-Piagetian theories have developed in an effort to define elusive structural concepts in terms of specific mental steps and strategies. This characterization makes Piagetian structures more amenable to precise definition and consequently to instructional intervention.

A number of attempts have been made to apply Piagetian theory to instruction. One approach is to teach Piagetian tasks as subject matter, in an effort to create the structures characteristic of operational thinking. Although successful training studies suggest this may be possible, the prospect of widespread teaching of this type is unlikely, given the high cost of extended training. The effort seems unwarranted for most children, because there is a high probability of their achiev-

ing at least concrete operational thinking through normal schooling and other experience. The development of formal operational thinking is less certain and therefore potentially more important for instruction. A few studies have shown that elementary school children can engage in formal operational thinking with appropriate training, but no test of this idea across a large array of tasks, as in a curriculum, has yet been undertaken.

A third instructional approach is to attempt to match instruction to children's developmental level. Rather than waiting for children to be "ready" for instruction, a more positive approach is to give children tasks that present something of an intellectual challenge but that have enough familiar elements so that they are comprehensible. General principles of constructive learning, concrete representations, social feedback, and clinical teacher-pupil interaction are derived from Piagetian theory. These can help to create optimal matches between learner capabilities and instructional content and procedures. We argue, however, that mathematics instruction is likely to benefit most from detailed psychological analysis of the content of the mathematics curriculum itself.

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